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LETTER TO THE EDITOR

Finest grading of the Lie superalgebra gl(n/n)

Joris Van der Jeugt

Department of Mathematics, University of Southampton, Southampton SO9 5NH, UK

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Abstract. We show that the group $\mathcal{P}_{2n}$ of generalised Pauli matrices in $2n$ dimensions provides a finest grading of the Lie superalgebra gl(n/n) and of its subalgebras sl(n/n) and A($n-1$, $n-1$).

In this letter we generalise some results obtained recently in the context of gradings for the Lie algebras gl(n, $\mathbb{C}$) (Patera and Zassenhaus 1987, 1988, Patera 1988) to the case of the Lie superalgebras gl(n/n). Although the Cartan decomposition with its grading by means of root spaces is very familiar and well known, both in the case of Lie algebras and Lie superalgebras, the question of other gradings for Lie algebras has only recently been studied, and many questions remain unsolved. Gradings for Lie (super)algebras are important because they give rise to preferred bases of the algebra which admit 'additive quantum numbers'. For the case of gl(n, $\mathbb{C}$) a simple solution to the problem of finding a finest grading was given in terms of the so-called generalised Pauli matrices. The set $\mathcal{P}_n$ of such Pauli matrices forms a subgroup of SL(n, $\mathbb{C}$), and provides a finest grading of the Lie algebra gl(n, $\mathbb{C}$) at the same time (Patera and Zassenhaus 1988).

Before applying similar ideas to the case of Lie superalgebras, let us recall some definitions and fix the notation. A Lie superalgebra L is a $\mathbb{Z}_2$-graded algebra $\mathrm{L} = \mathrm{L}_{\bar{0}} \oplus \mathrm{L}_{\bar{1}}$, for which the product (denoted by a bracket) satisfies

$$\begin{aligned} &[L_\alpha, L_\beta]/4 \subseteq L_{\alpha+\beta} && \alpha, \beta, \alpha+\beta \in \mathbb{Z}_2 \\ &[a, b] = -(-1)^{\alpha\beta}[b, a] && \\ &[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]] && a \in L_\alpha \qquad b \in L_\beta. \end{aligned} \tag{1}$$

$\mathrm{L}_{\bar{0}}$ is called the even subspace, $\mathrm{L}_{\bar{1}}$ is the odd subspace. The Lie superalgebra $\mathrm{L} = \mathrm{gl}(m/n)$ is the direct sum of two vector spaces $\mathrm{L}_{\bar{0}} \oplus \mathrm{L}_{\bar{1}}$, with

$$\begin{aligned} \mathrm{L}_{\bar{0}} &= \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right); A = m \times m \text{ matrix}, D = n \times n \text{ matrix in } \mathbb{C} \right\} \\ \mathrm{L}_{\bar{1}} &= \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right); B = m \times n \text{ matrix}, C = n \times m \text{ matrix in } \mathbb{C} \right\} \end{aligned} \tag{2}$$

where the bracket is defined in terms of the usual matrix product by

$$[a, b] = ab - (-1)^{\alpha\beta} ba \qquad a \in L_\alpha \qquad b \in L_\beta. \tag{3}$$

With the definition of supertrace as

$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D) \tag{4}$$

this product satisfies the property $\mathrm{str}[a, b] = 0$, and therefore the subspace

$$\mathrm{sl}(m/n) = \{a \in \mathrm{gl}(m/n) | \mathrm{str}(a) = 0\} \tag{5}$$

forms a subalgebra. The Lie superalgebra $\mathrm{sl}(m/n)$ is simple provided $m \neq n$, otherwise it contains the ideal $\mathbb{C}.I_{2n}$, where I_{2n} is the identity matrix in $2n$ dimensions. Then the quotient algebra $\mathrm{A}(n-1, n-1) = \mathrm{sl}(n/n)/\mathbb{C}.I_{2n}$ is simple (Kac 1977).

A *grading* of the Lie superalgebra L means that L can be written as a direct sum of linear subspaces

$$\mathrm{L} = X_\alpha \oplus X_\beta \otimes X_\gamma \oplus \ldots \qquad \alpha, \beta, \gamma \in S \tag{6}$$

labelled by a set S of finite sequences of integers or integers modulo k ($k \in \mathbb{N}$), such that

$$[X_\alpha, X_\beta] \subseteq X_{\alpha+\beta} \qquad \alpha, \beta, \alpha+\beta \in S. \tag{7}$$

The original $\mathbb{Z}_2$ grading of L satisfies this property, and is called the *supergrading*. A grading (6) is called *consistent* with the supergrading of L if every subspace X_α is either a subspace of $\mathrm{L}_{\bar{0}}$ or else a subspace of $\mathrm{L}_{\bar{1}}$. A consistent grading of L implies, of course, a $\mathbb{Z}_2$ grading for S. Let us illustrate this by means of a simple example. Let

$$\mathrm{L}_{\bar{0}} = \mathbb{C}a \qquad \mathrm{L}_{\bar{1}} = \mathbb{C}b \oplus \mathbb{C}c \tag{8}$$

with the only non-vanishing brackets among the basis $\{a, b, c\}$ given by

$$[b, c] = [c, b] = a. \tag{9}$$

Then

$$\mathrm{L} = X_0 \oplus X_1 \oplus X_2 \tag{10}$$

with $X_0 = \mathbb{C}a$, $X_1 = \mathbb{C}b$, $X_2 = \mathbb{C}c$, is a $\mathbb{Z}_3$ grading of L since $[X_\alpha, X_\beta] \subseteq X_{\alpha+\beta}$ ($\forall \alpha, \beta \in \mathbb{Z}_3$). Moreover, this $\mathbb{Z}_3$ grading is consistent with the supergrading. On the other hand,

$$\mathrm{L} = Y_0 \oplus Y_1 \oplus Y_2 \tag{11}$$

with $Y_0 = \mathbb{C}a$, $Y_1 = \mathbb{C}(a+b)$, $Y_2 = \mathbb{C}(a+c)$, is again a $\mathbb{Z}_3$ grading of L but now inconsistent with the supergrading.

Finally, a grading (6) is called a *finest grading* if every non-zero subspace X_α is one dimensional.

In order to give a finest grading for the Lie superalgebras $\mathrm{gl}(n/n)$, we recall the definition of the group $\mathcal{P}_{2n}$ of $2n \times 2n$ matrices with determinant 1. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & & 1 \\ -1 & 0 & & & \ldots & 0 \end{pmatrix} \tag{12}$$

and

$$D = \mathrm{diag}(\eta, \eta^3, \eta^5, \ldots, \eta^{4n-1}) \qquad \eta = \exp(2\pi \mathrm{i}/4n). \tag{13}$$

Clearly

$$A^{2n} = D^{2n} = -I_{2n}. \tag{14}$$

The group $\mathscr{P}_{2n}$ of generalised Pauli matrices consists of $(2n)^3$ matrices

$$K_{kad} = \eta^{2k} A^a D^d \qquad k, a, d \in \mathbb{Z}_{2n}. \tag{15}$$

Patera and Zassenhaus (1988) have shown that the $(2n)^2$ elements

$$\mathbb{K}_{ad} = K_{0ad} \qquad a, d \in \mathbb{Z}_{2n} \tag{16}$$

form a basis of $\mathrm{gl}(2n, \mathbb{C})$ and moreover provide a finest grading of the Lie algebra $\mathrm{gl}(2n, \mathbb{C})$.

Consider now the following $2n \times 2n$ matrix $X = (x_{ij})$ with entries 0 and 1 defined by

$$\begin{array}{ll} x_{ii} = i \bmod 2 & 1 \leqslant i \leqslant n \\ x_{ii} = (i+1) \bmod 2 & n+1 \leqslant i \leqslant 2n \\ x_{i,2n-i} = (i+1) \bmod 2 & 1 \leqslant i \leqslant n \\ x_{i,2n-i} = i \bmod 2 & n+1 \leqslant i \leqslant 2n \\ x_{ij} = 0 & \text{elsewhere.} \end{array} \tag{17}$$

Note that X is in fact a permutation matrix, and $X^2 = I_{2n}$. The similarity transforms of A, D and $\mathbb{K}_{ad}$ are denoted by

$$\begin{array}{ll} \tilde{A} = XAX & \tilde{D} = XDX \\ \tilde{\mathbb{K}}_{ad} = X\mathbb{K}_{ad}X = \tilde{A}^a \tilde{D}^d & a, d \in \mathbb{Z}_{2n}. \end{array} \tag{18}$$

With $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the explicit form of $\tilde{A}$ is

$$\left(\begin{array}{cccc|cccc} & & & & & & & I_2 \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & I_2 & & \\ & & & & 1 & & & \\ \hline & & & I_2 & & & & \\ & & \cdot^{\cdot^{\cdot}} & & & & & \\ & I_2 & & & & & & \\ -1 & & & & & & & \end{array}\right) \quad \text{for } n \text{ odd}$$

$$\left(\begin{array}{cccc|cccc} & & & & & & & I_2 \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & I_2 & & \\ & & & & I_2 & & & \\ \hline & & & 1 & & & & \\ & & I_2 & & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & & \\ I_2 & & & & & & & \\ -1 & & & & & & & \end{array}\right) \quad \text{for } n \text{ even} \tag{19}$$

and

$$\begin{aligned}\tilde{D} &= \mathrm{diag}(\eta, \eta^{4n-3}, \eta^5, \eta^{4n-7}, \ldots, \eta^{2n-5}, \eta^{2n+3}; \eta^{2n-1}, \eta^{2n+1};\\ &\qquad \eta^{2n-3}, \eta^{2n+5}, \eta^{2n-7}, \eta^{2n+9}, \ldots, \eta^3, \eta^{4n-1}) \qquad \text{for } n \text{ odd}\\ \tilde{D} &= \mathrm{diag}(\eta, \eta^{4n-3}, \eta^5, \eta^{4n-7}, \ldots, \eta^{2n-3}, \eta^{2n+1};\\ &\qquad \eta^{2n-1}, \eta^{2n+3}, \eta^{2n-5}, \eta^{2n+7}, \ldots, \eta^3, \eta^{4n-1}) \qquad \text{for } n \text{ even.}\end{aligned} \tag{20}$$

The matrices $\tilde{\mathbb{K}}_{ad}$ span the vector space of $\mathrm{gl}(2n, \mathbb{C})$; hence they also span the vector space of $\mathrm{gl}(n/n)$. Moreover, from (19) and (20) it follows that $\tilde{A} \in \mathrm{gl}(n/n)_{\bar{1}}$ and $\tilde{D} \in \mathrm{gl}(n/n)_{\bar{0}}$, and hence the subspaces $\mathrm{gl}(n/n)_\alpha$ $(\alpha \in \mathbb{Z}_2)$ are spanned by

$$\begin{aligned}\mathrm{gl}(n/n)_{\bar{0}} &= \mathrm{span}\{\tilde{\mathbb{K}}_{ad}, a \text{ even}\}\\ \mathrm{gl}(n/n)_{\bar{1}} &= \mathrm{span}\{\tilde{\mathbb{K}}_{ad}, a \text{ odd}\}.\end{aligned} \tag{21}$$

Therefore

$$\mathrm{gl}(n/n) = \bigoplus_{(a,d) \in \mathbb{Z}_{2n}^2} \mathbb{C} \cdot \tilde{\mathbb{K}}_{ad} \tag{22}$$

and this forms a grading of $\mathrm{gl}(n/n)$ provided condition (7) is satisfied. But the $\tilde{\mathbb{K}}_{ad}$ matrices satisfy

$$\tilde{\mathbb{K}}_{ad} \cdot \tilde{\mathbb{K}}_{a'd'} = \varepsilon \eta^{-2a'd} \tilde{\mathbb{K}}_{a+a', d+d'} \tag{23}$$

where $\varepsilon = -1$ if $0 \leqslant a + a' < 2n \leqslant d + d' < 4n$ or $0 \leqslant d + d' < 2n \leqslant a + a' < 4n$ and $\varepsilon = 1$ otherwise (the minus sign appears because of the minus sign in $\tilde{A}^{2n} = \tilde{D}^{2n} = -I_{2n}$). Consequently the Lie superalgebra bracket is given by

$$[\tilde{\mathbb{K}}_{ad}, \tilde{\mathbb{K}}_{a'd'}] = \varepsilon(\eta^{-2a'd} - (-1)^{a+a'} \eta^{-2ad'}) \tilde{\mathbb{K}}_{a+a', d+d'} \tag{24}$$

and thus (22) is indeed a grading of $\mathrm{gl}(n/n)$. Moreover it is a finest grading since every subspace $\mathbb{C} \cdot \tilde{\mathbb{K}}_{ad}$ is one dimensional. Hence we have shown that the generalised Pauli matrices, besides providing a finest grading for the Lie algebra $\mathrm{gl}(n, \mathbb{C})$, can be transformed in order to provide a finest grading for the Lie superalgebra $\mathrm{gl}(n/n)$. A similar technique for $\mathrm{gl}(m/n)$ $(m \neq n)$ cannot work since there is no permutation matrix X that transforms A into an odd matrix of $\mathrm{gl}(m/n)$. We have tried different approaches to find a finest grading for $\mathrm{gl}(m/n)$ $(m \neq n)$, but all were unsuccessful—in fact it is an open question whether $\mathrm{gl}(m/n)$ with $m \neq n$ does in general possess a finest grading at all.

Let us finally give some of the extra properties of the matrices defining the finest grading of $\mathrm{gl}(n/n)$. From the explicit form of $\tilde{D}$, one can calculate that

$$\begin{aligned}\mathrm{str}(\tilde{D})^k &= 0 \qquad \text{for } k \neq n \qquad 0 \leqslant k < 2n.\\ \mathrm{str}(\tilde{D})^n &= 2n\mathrm{i}.\end{aligned} \tag{25}$$

Hence $\mathrm{sl}(n/n)$ is spanned by the matrices $\tilde{\mathbb{K}}_{ad}$ with $(a, d) \neq (0, n)$. It can also be verified from the Lie superalgebra bracket that $\tilde{K}_{0n}$ never appears on the right-hand side of (24), leading to the same conclusion. Obviously $\tilde{\mathbb{K}}_{00} = I_{2n}$, and so the simple Lie superalgebra $\mathrm{A}(n-1, n-1)$ is spanned by

$$\mathrm{A}(n-1, n-1) = \bigoplus_{\substack{(a,d) \in \mathbb{Z}_{2n}^2\\ (a,d) \neq (0,n)\\ (a,d) \neq (0,0)}} \mathbb{C} \cdot \tilde{\mathbb{K}}_{ad} \tag{26}$$

where (26) is a finest grading of $\mathrm{A}(n-1, n-1)$.

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